Esercizi
(per sostituzione)

1) $$\int \frac{1}{1+e^{2x}}\,dx = \int\left(\frac{1}{1+t}\right)\frac{1}{2t}\,dt = \frac{1}{2}\int\frac{1}{t(t+1)}\,dt$$

$$\left[\begin{array}{l} t=e^{2x} \Rightarrow x=\frac{\ln t}{2} \\ \qquad\qquad dx=\frac{1}{2t}\,dt \end{array}\right] \qquad \frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{t+1}$$

$$= \frac{1}{2}\int\frac{1}{t}\,dt - \frac{1}{2}\int\frac{1}{t+1}\,dt = \frac{1}{2}\ln(e^{2x}) - \frac{1}{2}\ln(e^{2x}+1)+c$$

$$= \frac{1}{2}\ln\left(\frac{e^{2x}}{e^{2x}+1}\right)+c$$

2) $$\int_1^{e^3}\frac{\sqrt{1+\ln x}}{x}\,dx$$

$$\left[\begin{array}{l} u=1+\ln x \\ du=\frac{1}{x}\,dx \end{array}\right] \qquad \int \overset{u^{1/2}}{\sqrt{u}}\,du = \frac{2}{3}u^{3/2}+c = \frac{2}{3}(1+\ln x)^{3/2}+c$$

$$\int_1^{e^3}\frac{\sqrt{1+\ln x}}{x}\,dx = \frac{2}{3}(1+\ln x)^{3/2}\Big|_1^{e^3} = \frac{2}{3}\Big[\underbrace{(1+\ln e^3)^{3/2}}_{(\ln e+\ln e^3)^{3/2}} - \underbrace{(1+\underbrace{\ln 1}_{0})^{3/2}}_{1}\Big] = \frac{2}{3}(8-1) = \frac{14}{3}$$

$$=(\ln e\cdot e^3)^{3/2}$$
$$=(\ln e^4)^{3/2} = 4^{3/2} = 8$$

3) $$\int xe^{x^2}\,dx = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^{x^2}+c$$

$$\left[\begin{array}{l} u=x^2 \\ du=2x\,dx \end{array}\right]$$

4) $$I=\int \cos^4 x\sin^5 x\,dx = \int\cos^4 x\sin^4 x\sin x\,dx$$

[Bisogna porre $u=\cos x$ (si sceglie quella con potenza pari)
$du=-\sin x\,dx$]

$$\sin^4 x = (\sin^2 x)^2 = (1-\cos^2 x)^2 = 1-2\cos^2 x+\cos^4 x$$

$$I = -\int \underbrace{u^4(1-2u^2+u^4)}_{u^4-2u^6+u^8}\,du$$

$$= -\frac{u^5}{5}+2\frac{u^7}{7}-\frac{u^9}{9}+c$$

$$= -\frac{\cos^5 x}{5}+2\frac{\cos^7 x}{7}-\frac{\cos^9 x}{9}+c$$

5) $\int \cos^2 x\,dx$

$$\boxed{\cos^2 x = \frac{1+\cos 2x}{2} \quad ; \quad \sin^2 x = \frac{1-\cos 2x}{2}}$$

$$\int \cos^2 x\,dx = \frac{1}{2}\int (1+\cos 2x)\,dx = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

6) $I = \int \sin^6 x\,dx$

$$\sin^6(x) = (\sin^2 x)^3 = \left(\frac{1-\cos 2x}{2}\right)^3 = \frac{1}{8}(1-\cos 2x)^3$$

$$= \frac{1}{8}\left(1 - 3\cos 2x + 3\cos^2 2x - \cos^3 2x\right)$$

$$= \frac{1}{8}\left(1 - 3\cos 2x + 3\left(\frac{1+\cos(4x)}{2}\right) - \cos^3 2x\right)$$

$$= \frac{1}{8}\left(\frac{2 - 6\cos 2x + 3 + 3\cos(4x)}{2} - \cos^3 2x\right)$$

$$= \frac{1}{16}\left(5 - 6\cos(2x) + 3\cos(4x)\right) - \frac{1}{8}\cos^3 2x$$

$$\int \sin^6 x\,dx = \frac{1}{16}\left(5x - 3\sin(2x) + \frac{3}{4}\sin(4x)\right) - \frac{1}{8}\underbrace{\int \cos^3(2x)\,dx}_{I_2}$$

$$I_2 = \int \cos^3(2x)\,dx = \int \cos^2(2x)\cos(2x)\,dx = \int (1-\sin^2(2x))\cos(2x)\,dx$$

$$= \int \cos(2x)\,dx - \int \sin^2(2x)\cos(2x)\,dx = \frac{\sin 2x}{2} - \frac{1}{2}\int u^2\,du$$

$$\begin{bmatrix} u = \sin(2x) \\ du = 2\cos(2x)\,dx \end{bmatrix} \qquad = \frac{\sin 2x}{2} - \frac{1}{2}\,\frac{\sin^3(2x)}{3} = \frac{1}{2}\left(\sin 2x - \frac{\sin^3(2x)}{3}\right)$$

$$I = \frac{1}{16}\left(5x - 3\sin(2x) + \frac{3}{4}\sin(4x)\right) - \frac{1}{16}\left(\sin(2x) - \frac{\sin^3(2x)}{3}\right) + C$$

$$I = \frac{1}{16}\left(5x - 4\sin(2x) + \frac{3}{4}\sin(4x) + \frac{\sin^3(2x)}{3}\right) + C$$

(Funz. razionali)

7) $I = \int \frac{x^4 + 2x^2}{x^2 - 1}\,dx$

$$\begin{array}{r|l} x^4 + 2x^2 & x^2 - 1 \\ \hline -x^4 + x^2 & x^2 + 3 \\ \hline 3x^2 & \\ -3x^2 + 3 & \\ \hline 3 & \end{array} \Rightarrow x^2 + 3 + \frac{3}{x^2-1}$$

$$I = \underbrace{\int (x^2+3)\,dx}_{I_1} + 3\underbrace{\int \frac{1}{x^2-1}\,dx}_{I_2}$$

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1}_{(x+1)} + \frac{B}{x+1}_{(x-1)}$$

$$Ax + A + Bx - B = 1$$

$$\left.\begin{array}{l} A + B = 0 \\ A - B = 1 \end{array}\right\} \; 2A = 1 \Rightarrow A = 1/2, \quad B = -1/2$$

$$I_2 = \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{1}{x+1}\,dx = \frac{1}{2}\left(\ln|x-1| - \ln|x+1|\right)$$

$$I = \frac{x^3}{3} + 3x + \frac{3}{2}\ln\left|\frac{x-1}{x+1}\right| + c$$

8) $$I = \int \frac{6x}{4x^2+12x+9}\,dx \qquad 4x^2+12x+9 = (2x+3)^2$$

$$\begin{matrix} 2x & 3 \\ 2x & 3 \end{matrix}$$

$$\frac{6x}{(2x+3)^2} = \frac{A}{2x+3} + \frac{B}{(2x+3)^2} \Rightarrow \begin{matrix} A=3 \\ B=-9 \end{matrix}$$

$$I = 3\int \frac{1}{2x+3}\,dx - 9\int \frac{1}{(2x+3)^2}\,dx = 3\,\frac{\ln|2x+3|}{2} + \frac{9}{2}\cdot\frac{1}{(2x+3)} + c$$

$$= \frac{3}{2}\left(\ln|2x+3| + \frac{3}{2x+3}\right) + c$$

9) $$\int \frac{13}{x^2-4x+8}\,dx = 13\int \frac{1}{\underbrace{(x-2)^2}_{x^2-4x+4}+\underbrace{4}_{a=2}} = 13\cdot\frac{1}{2}\tan^{-1}\left(\frac{x-2}{2}\right) + c$$

(per parti)

10) $$\int_0^1 x e^{3x}\,dx = \frac{1}{3}x e^{3x} - \frac{1}{3}\int_0^1 e^{3x}\,dx = \frac{1}{3}x e^{3x} - \frac{1}{3}\cdot\frac{e^{3x}}{3}\Bigg|_0^1 = \frac{1}{3}\left(x e^{3x} - \frac{e^{3x}}{3}\right)\Bigg|_0^1$$

$$\left[\begin{matrix} u=x & dv = e^{3x}dx \\ du=dx & v=\frac{e^{3x}}{3} \end{matrix}\right]$$

$$= \frac{1}{3}\left[\left(1\cdot e^3 - \frac{1}{3}e^3\right) - \left(0 - e^0\right)\right] = \frac{1}{3}\left(\frac{2}{3}e^3 + 1\right)$$

$$= \frac{2}{9}e^3 + \frac{1}{3}$$

11) $$\int \ln x\,dx = x\ln x - \int 1\cdot dx = x\ln x - x + c$$

$$\left[\begin{matrix} u=\ln x & dv=dx \\ du=\frac{1}{x}dx & v=x \end{matrix}\right]$$

$$\int \frac{dx}{a^2-x^2} = -\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$$

(Improprio)

12) $$\int_2^3 \frac{8}{4-x^2}\,dx = \lim_{c\to 2^+}\int_c^3 \frac{8}{4-x^2}\,dx = \lim_{c\to 2^+}\left(-\frac{\cancel{8}^{\,2}}{\cancel{4}}\ln\left|\frac{x-2}{x+2}\right|\right)\Bigg|_c^3$$

$(a=2)$

$$\lim_{c\to 2^+}\ln\left|\frac{x+2}{x-2}\right|^2\Bigg|_c^3 = \ln 5^2 - \underbrace{\lim_{c\to 2^+}\ln\left|\frac{c+2}{c-2}\right|^2}_{+\infty} = 2\ln 5 - \infty = -\infty$$

Quindi l'integrale improprio di partenza diverge a $-\infty$.

13) $\displaystyle\int_3^{+\infty} \frac{1}{x\ln^2 x}\,dx = \lim_{R\to+\infty} \int_3^{R} \frac{1}{x\ln^2 x}\,dx = \lim_{R\to+\infty} -\frac{1}{\ln x}\Big|_3^R$

$$\begin{bmatrix} u=\ln x \\ du=\frac{1}{x}dx \end{bmatrix} \qquad = \lim_{R\to+\infty} -1\left(\overset{\nearrow 0}{\frac{1}{\ln R}} - \frac{1}{\ln 3}\right)$$

$$\int \frac{1}{x\ln^2 x}\,dx = \int u^{-2}\,du = -\frac{1}{u}+c \qquad = \frac{1}{\ln 3}$$

Quindi l'integrale improprio di potenza converge

(serie e succ. di funz.)

14) Studiare la convergenza puntuale di $\{f_n\} = \{2^{-nx}\}$ in $\mathbb{R}$.

$$\lim_{n\to\infty} 2^{-nx} = f(x) = \begin{cases} 0 & \text{se } x>0 \\ 1 & \text{se } x=0 \end{cases}$$

quindi f_n conv. puntualmente ad f in $[0,+\infty)$, ma <u>non</u> conv. punt. in $(-\infty,0)$. Infatti;

se $x<0 \Rightarrow \lim_{n\to\infty} 2^{-nx} = +\infty$

15) Studiare la convergenza uniforme di

$$\sum_{k=1}^{+\infty} \frac{x}{x^2+k^3} \quad \text{in } \mathbb{R}.$$

$f_k(x) = \dfrac{x}{x^2+k^3}$, consideriamo la funzione $g_n(x) = \dfrac{|x|}{x^2+n^2}$, $n>0$

(dove $n^2=k^3$, ossia $n=k^{3/2}$)

Dato che g è una funzione pari, consideriamo per $x\geq 0$, abbiamo la derivata:

$$g_n'(x) = \frac{x^2+n^2-x(2x)}{(x^2+n^2)^2} = \frac{n^2-x^2}{(x^2+n^2)^2} = 0 \Rightarrow n^2-x^2=0$$
$$x^2=n^2 \Rightarrow x=n,\ x\geq 0$$

Perciò, $\max_{\mathbb{R}} g_n(x) = g_n(n) = \dfrac{n}{2n^2} = \dfrac{1}{2n}$

Per $n=k^{3/2}$ si ottiene $M_k = \max_{\mathbb{R}} \dfrac{|x|}{x^2+k^3} = \dfrac{1}{2k^{3/2}}$ e $|f_k(x)| \leq M_k$

Sappiamo che $\sum \frac{1}{k^{3/2}}$ è una serie armonica generalizzata con $p=3/2>1$ che converge

quindi la serie $\sum f_k$ converge totalmente, quindi converge anche uniformemente!

(serie formali di potenze e succ. definite per ricorrenza)

16) Data $\{a_n\}$, $n \geq 0$ con $a_n = 5 \cdot 7^n - 3 \cdot 4^n$ scrivere la funzione generatrice associata.

$$\frac{1}{1-ax} = \sum a^n x^n$$

$$a_n = 5 \cdot 7^n - 3 \cdot 4^n$$

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = 5 \sum_{n=0}^{\infty} 7^n x^n - 3 \sum_{n=0}^{\infty} 4^n \cdot x^n$$

$$= \frac{5}{1-7x} - \frac{3}{1-4x}$$

$$= \frac{5-20x-3+21x}{(1-7x)(1-4x)} = \frac{2+x}{(1-7x)(1-4x)}$$

17) per $\{a_n\}$, $n \geq 0$ con $a_n = n$

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n x^n = \cancel{0 \cdot x^0} + 1 \cdot x^1 + 2 \cdot x^2 + 3x^3 + \dots + nx^n + \dots$$

$$= x(1 + 2x + 3x^2 + \dots + nx^{n-1} + \dots)$$

Sappiamo che

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad \text{e} \quad \underbrace{\frac{d}{dx}\left(\frac{1}{1-x}\right)}_{\frac{1}{(1-x)^2}} = 1 + 2x + 3x^2 + \dots$$

Quindi $A(x) = x \cdot \frac{1}{(1-x)^2}$

Risolvere l'equazione differenziale

18) $$y' = \frac{x}{y-3} \Rightarrow \frac{dy}{dx} = \frac{x}{y-3}$$

$$\int (y-3)\, dy = \int x\, dx$$

$$\frac{y^2}{2} - 3y = \frac{x^2}{2} + c$$

$$y^2 - 6y - x^2 + c = 0$$

19) $$\begin{cases} x^2 y' + y(1-x) = 0 \\ y(1) = 1 \end{cases}$$ Problema di Cauchy

$$x^2 \frac{dy}{dx} = (x-1)y$$

$$\frac{1}{y}\,dy = \frac{x-1}{x^2}\,dx$$

$$\int \frac{1}{y}\,dy = \int \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$$

$$\ln|y| = \ln|x| + \frac{1}{x} + c$$

$$|y| = e^{\ln|x| + 1/x + c}$$

$$|y| = |x| \cdot e^{1/x + c}$$

$$1 = 1 \cdot e^{1+c} \Rightarrow \ln 1 = \ln e^{1+c} \Rightarrow 1 + c = 0 \Rightarrow c = -1$$

Quindi,

$$|y| = |x| \cdot e^{1/x - 1}$$

(non omogenea)

20) $y' = \frac{1}{x} y - \frac{\ln x}{x}$; $y(e) = 2$ Prob. di Cauchy

$a(x) = \frac{1}{x}$; $b(x) = -\frac{\ln x}{x}$

$$y = e^{\int \frac{1}{x} dx} \left[-\int \frac{\ln x}{x} \cdot e^{-\int \frac{1}{x} dx} dx + c \right]$$

$$y = \underbrace{e^{\ln x}}_{x} \left[-\int \frac{\ln x}{x} \underbrace{e^{-\ln x}}_{x^{-1}} dx + c \right]$$

$$\int \frac{\ln x}{x^2}\,dx = -\frac{1}{x}\ln x + \int \frac{1}{x^2}\,dx = -\frac{1}{x}\ln x - \frac{1}{x} + c$$

x^{-2}

$$\left[\begin{array}{ll} u = \ln x & dv = \frac{1}{x^2}dx \\ du = \frac{1}{x}dx & v = -\frac{1}{x} \end{array}\right]$$

$$y = x\left(\frac{1}{x}\ln x + \frac{1}{x} + c\right)$$

$$y = \ln x + 1 + xc$$

$$2 = \underbrace{\ln e}_{1} + 1 + ec \Rightarrow c = 0$$

Quindi;

$$\boxed{y = \ln x + 1}$$