Soluzioni di Compitino 2
online

5

1 $\lim_{n\to\infty} \frac{2n+1}{n+5} = 2$

$\forall \varepsilon > 0,\ \exists N(\varepsilon)$ tale che $\forall n > N,\ \left|\frac{2n+1}{n+5} - 2\right| < \varepsilon$

$$\underbrace{2-\varepsilon < \frac{2n+1}{n+5}}_{1^\circ} \overbrace{< 2+\varepsilon}^{2^\circ}$$

$1^\circ)$ $2n+10-\varepsilon n - 5\varepsilon < 2n+1$

$\varepsilon n < 9 - 5\varepsilon$

$n > \frac{9-5\varepsilon}{\varepsilon}$

$2^\circ)$ $\cancel{2n}+1 < \cancel{2n}+10+\varepsilon n + 5\varepsilon$

$\varepsilon n > -9-5\varepsilon$

$n > \frac{-9-5\varepsilon}{\varepsilon}$

<0 sempre soddisfatto

Quindi basta scegliere $\boxed{N = \frac{9-5\varepsilon}{\varepsilon}}$

5 2 LN: $\lim_{n\to\infty} \frac{\sin a_n}{a_n} = 1$ dove $a_n = \frac{1}{2n} \to 0$ per $n\to\infty$

$\Rightarrow \boxed{\sin\frac{1}{2n} \sim \frac{1}{2n}}$

$$e^{\frac{1}{n^2 \sin(1/2n)}} \sim e^{\frac{1}{n^2\cdot\frac{1}{2n}}} = e^{\frac{2}{n}}$$

$$\lim_{n\to\infty} n\left(e^{\frac{1}{n^2\sin(1/2n)}} - 1\right) = \lim_{n\to\infty} n\cdot\left(e^{2/n} - 1\right)$$

LN: $\boxed{e^{2/n} - 1 \sim 2/n}$ perché $\lim_{n\to\infty} \frac{e^{2/n}-1}{2/n} = 1$, $\frac{2}{n}\to 0$ per $n\to\infty$

$$= \lim_{n\to\infty} \cancel{n}\cdot\frac{2}{\cancel{n}} = 2$$

(3) $\lim_{n\to\infty} \dfrac{(1-e^{1/n})(1-\cos(\frac{1}{n}))\cdot n^3}{\sin^2(1/n)}$

LN: $e^{1/n}-1 \sim \frac{1}{n} \Rightarrow 1-e^{-\frac{1}{n}} \sim -\frac{1}{n}$

LN: $1-\cos(1/n) \sim \frac{1}{2}\cdot\left(\frac{1}{n}\right)^2$

LN: $\sin\left(\frac{1}{n}\right) \sim \frac{1}{n}$

Quindi abbiamo

$$\lim_{n\to\infty} \frac{\left(-\frac{1}{n}\right)\frac{1}{2}\cancel{\left(\frac{1}{n^2}\right)}\cdot n^3}{\cancel{\frac{1}{n^2}}} = \lim_{n\to\infty} -\frac{1}{2}\cdot n^2 = -\infty$$

(6) (4) (a) $\sum_{n=4}^{\infty}\left(\frac{n}{2n+1}\right)^n$

Per il criterio della radice:

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{2n+1}\right)^n} = \lim_{n\to\infty}\frac{n}{2n+1} = \frac{1}{2} < 1$$

Quindi $\sum_{n=4}^{\infty}\left(\frac{n}{2n+1}\right)^n$ converge

(b) $\sum_{n=1}^{\infty}\frac{n^n}{\pi^n\cdot n!}$

Per il criterio del rapporto

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+1)^{n+1}}{\pi^{n+1}\cdot(n+1)!}\cdot\frac{\pi^n\cdot n!}{n^n} = \lim_{n\to\infty}\frac{(n+1)^n\cancel{(n+1)}\cdot\cancel{\pi^n}\cdot\cancel{n!}}{\cancel{\pi^n}\cdot\pi\,\cancel{(n+1)}\,\cancel{n!}\,n^n}$$

$$= \frac{1}{\pi}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \frac{1}{\pi}\underbrace{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}_{e} = \frac{e}{\pi} \sim \frac{2.17}{3.14} < 1 \Rightarrow \text{converge}$$

5 5 $$\sum_{n=1}^{\infty} \frac{\sin(2n+1)\,\pi/2}{n^2}$$

$$\sin\left((2n+1)\frac{\pi}{2}\right) = \sin\left(n\pi + \frac{\pi}{2}\right) = \{-1, 1, -1, 1, \dots\}$$

$$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n^2}$$

$\left|\frac{(-1)^n}{n^2}\right| = \frac{1}{n^2}$, $\sum \frac{1}{n^2}$ è una serie armonica generalizzata con $p>1$, convergente

Quindi la serie di partenza converge assolutamente

4 6 $$\sum_{n=1}^{\infty} 4 \cdot \left(\frac{1}{5}\right)^{n-1}$$ converge perchè una serie geometrica con $|r| = \left|\frac{1}{5}\right| < 1$

Questa serie converge a $4 \cdot \frac{1}{1-\frac{1}{5}} = 4 \cdot \frac{1}{\frac{4}{5}} = 5$